

# CHARACTERISTIC VARIETIES OF NILPOTENT GROUPS AND APPLICATIONS

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**ABSTRACT.** We compute the characteristic varieties and the Alexander polynomial of a finitely generated nilpotent group. We show that the first characteristic variety may be used to detect nilpotence. We use the Alexander polynomial to deduce that the only torsion-free, finitely generated nilpotent groups with positive deficiency are  $\mathbb{Z}$  and  $\mathbb{Z}^2$ , extending a classical result on nilpotent link groups.

## 1. INTRODUCTION

Let  $M$  be a connected CW-complex with finite 1-skeleton. Let  $\mathbb{T}_M$  be the algebraic group  $\mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ , the *character torus* of  $M$ . We denote by  $\mathbb{C}_\rho$  the rank one complex local system on  $M$  corresponding to a character  $\rho \in \mathbb{T}_M$ . The *characteristic varieties*  $\mathcal{V}_k^i(M)$  are defined by

$$(1.1) \quad \mathcal{V}_k^i(M) = \{\rho \in \mathbb{T}_M \mid \dim H^i(M, \mathbb{C}_\rho) \geq k\},$$

for  $i, k > 0$ . They emerged from Novikov's work [17] on Morse theory for circle-valued functions. Their importance was recognized in various other areas, and their study was vigorously pursued. See for instance [4, 5, 6, 7], where Serre's problem on fundamental groups of smooth complex algebraic varieties is attacked through the prism of the cohomology jumping loci from (1.1).

Given a finitely generated group  $G$ , one may replace  $M$  by the classifying space  $K(G, 1)$  in the above definitions (simply changing  $M$  to  $G$  in the notation). Here is our first result (also proved by Alaniya [1], by using Lie algebra techniques, but only for torsion-free groups  $G$  and characters  $\rho$  belonging to the identity component of  $\mathbb{T}_G$ ). See Theorem 2.8 and Example 2.6 for a more general statement.

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**Theorem 1.1.** *Let  $G$  be a finitely generated nilpotent group. Then*

$$\mathcal{V}_k^i(G) = \begin{cases} \{1\}, & \text{if } b_i(G) \geq k; \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is worth pointing out that computing twisted (co)homology is a very difficult task, in general. In degree one, the characteristic varieties  $\{\mathcal{V}_k^1(G)\}$  depend only on the metabelian quotient  $G/G''$ . In particular, there is a systematic way of producing solvable examples with pretty complicated characteristic varieties  $\mathcal{V}_1^1(G) \subseteq \mathbb{T}_G$ , by Fox calculus. See [7].

In degree one, Theorem 1.1 says that

$$(1.2) \quad \mathcal{V}_1^1(G) \subseteq \{1\},$$

for a finitely generated nilpotent group. A notable feature of property (1.2) is that it can distinguish nilpotence from solvability, for large classes of groups.

**Theorem 1.2.** *Let  $G$  be a finitely generated, torsion-free metabelian group, with torsion-free abelianization. Then  $G$  is nilpotent if and only if  $\mathcal{V}_1^1(G) \subseteq \{1\}$ .*

See also Theorem 2.10 for a similar result.

There is an infinitesimal analog of characteristic varieties, namely the so-called *resonance varieties*  $\{\mathcal{R}_k^i(M) \subseteq H^1(M, \mathbb{C})\}_{i,k>0}$ , defined in terms of the cohomology ring of  $M$ . In degree one, there is a completely contrasting resonance counterpart of Theorem 1.1: the resonance varieties  $\{\mathcal{R}_k^1(G)\}_k$  of a two-step nilpotent, torsion-free group  $G$ , can be as complicated as those of an arbitrary complex  $M$ . See Remark 2.4.

A finitely generated group  $G$  has *Alexander polynomial*  $\Delta^G \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ ,  $n = b_1(G)$ , well defined up to units of the Laurent polynomial ring, see e.g. [16]. As a second application of Theorem 1.1, we compute Alexander polynomials of nilpotent groups.

**Corollary 1.3.** *If  $G$  is a finitely generated nilpotent group,  $\Delta^G$  is a non-zero constant, up to units.*

A group  $G$  has *positive deficiency* (notation:  $\text{def}(G) > 0$ ) if it has a finite presentation with strictly more generators than relations. It is well-known that fundamental groups of link complements in  $S^3$  have positive deficiency, see e.g. [9]. Corollary 1.3 in turn provides the main step in deriving our third application of Theorem 1.1.

**Theorem 1.4.** *The only torsion-free, finitely generated nilpotent groups with positive deficiency are  $\mathbb{Z}$  and  $\mathbb{Z}^2$ .*

Taking into account that link groups are also torsion-free (as follows e.g. from [14]), one may view the above theorem as an extension of a classical result in low-dimensional topology: the only nilpotent link groups are  $\mathbb{Z}$  and  $\mathbb{Z}^2$ ; see for instance [10], [3], and the references therefrom.

Along the way, we extend another result in classical link theory, which says that the Alexander polynomial of the link group vanishes at 1, for links with at least 3 components; see Torres [21]. A similar result holds, not only for link groups but for all groups whose Alexander ideal is *almost principal*, in the sense of [7]. See Proposition 3.11.

## 2. JUMPING LOCI AND POLY-CYCLIC GROUPS

**2.1. Jump loci.** Due to the fact that  $\dim H^i(M, \mathbb{C}_1) = b_i(M)$ , Theorem 1.1 is a consequence of the following vanishing result.

**Theorem 2.2.** *For any finitely generated nilpotent group  $G$ , for all  $i \geq 0$  and  $\rho \neq 1$ ,  $H^i(G, \mathbb{C}_\rho) = 0$ .*

*Proof.* By duality, the statement amounts to proving the vanishing of twisted homology. This we do, by induction on the nilpotence class of  $G$ . If  $G$  is abelian, the claim follows easily for cyclic groups, by a direct computation using the standard resolution of  $\mathbb{Z}$  over  $\mathbb{Z}G$ , and then in general, by resorting to the Künneth formula. For the induction step, we use the Hochschild–Serre spectral sequence. See [2].

Recall from [2, p. 171] that, given a group extension,

$$(2.1) \quad 1 \rightarrow K \rightarrow G \xrightarrow{p} B \rightarrow 1,$$

and a  $G$ -module  $M$ , there is a spectral sequence,

$$(2.2) \quad E_{st}^2 = H_s(B, H_t(K, M)) \Rightarrow H_{s+t}(G, M).$$

If  $P_\bullet$  is a  $G$ -resolution of  $\mathbb{Z}$ , the  $B$ -action on  $H_*(K, M) = H_*(M \otimes_K P)$  is induced by the tensor product of the  $G$ -action on  $M$  and the  $G$ -action on  $P$ .

Assume that  $M = \mathbb{C}_\rho$ , with  $1 \neq \rho \in \mathbb{T}_G$ , and the  $G$ -action on  $M$  factors through  $B$ , that is,  $\rho = p^* \rho'$ , with  $1 \neq \rho' \in \mathbb{T}_B$ . Then  $\mathbb{C}_\rho$  is  $K$ -trivial, and  $H_*(K, M) = H_*(K, \mathbb{Z}) \otimes \mathbb{C}$ . The  $B$ -action on  $H_*(K, M)$  is then induced by the tensor product of the  $G$ -conjugation action on  $H_*(K, \mathbb{Z})$  and the  $G$ -action on  $\mathbb{C}$ . If moreover the extension (2.1) is central, we infer that  $H_*(K, \mathbb{C}_\rho)$  is a direct sum of copies of  $\mathbb{C}_{\rho'}$ , over  $B$ .

Consider now the central extension

$$(2.3) \quad 1 \rightarrow \Gamma_i G / \Gamma_{i+1} G \rightarrow G / \Gamma_{i+1} G \xrightarrow{p} G / \Gamma_i G \rightarrow 1,$$

with  $i \geq 2$ . Here  $\Gamma_j G$  are the  $j$ -fold commutators in  $G$ :  $\Gamma_1 G = G$ ,  $\Gamma_2 G = G' = (G, G)$ , and inductively  $\Gamma_j G = (G, \Gamma_{j-1} G)$ , where  $(,)$  stands for the group commutator.

Pick any  $1 \neq \rho \in \mathbb{T}_{G/\Gamma_{i+1}G}$ . Since  $i \geq 2$ ,  $\rho = p^* \rho'$ , with  $1 \neq \rho' \in \mathbb{T}_{G/\Gamma_i G}$ . By the above discussion,

$$E_{st}^2 = H_s(G/\Gamma_i G, \oplus \mathbb{C}_{\rho'}) = \oplus H_s(G/\Gamma_i G, \mathbb{C}_{\rho'}).$$

By induction,  $E_{st}^2 = 0$ , for all  $s, t$ . Hence,  $H_*(G/\Gamma_{i+1}G, \mathbb{C}_{\rho}) = 0$ , by (2.2).

Due to nilpotence,  $G/\Gamma_j G = G$ , for  $j$  large.  $\square$

Let  $M$  be a connected CW complex with finite 1-skeleton, as before. For  $z \in H^1(M, \mathbb{C})$ , denote by  $\mu_z$  left-multiplication by  $z$ , acting on  $H^\bullet(M, \mathbb{C})$ . Since  $z^2 = 0$ ,  $(H^\bullet(M, \mathbb{C}), \mu_z)$  is a cochain complex. The *resonance varieties*  $\mathcal{R}_k^i(M)$  are defined by

$$(2.4) \quad \mathcal{R}_k^i(M) = \{z \in H^1(M, \mathbb{C}) \mid \dim H^i(H^\bullet(M, \mathbb{C}), \mu_z) \geq k\},$$

for  $i, k > 0$ .

Let  $\cup_M: \wedge^2 H^1(M, \mathbb{Q}) \rightarrow H^2(M, \mathbb{Q})$  be the cup-product. Denote by  $K_M$  the kernel of  $\cup_M$ , and by  $\mu_M: \wedge^2 H^1(M, \mathbb{Q}) \twoheadrightarrow DH_M^2$  the corestriction of  $\cup_M$  to its image. Plainly, the resonance varieties  $\mathcal{R}_k^1(M)$  depend only on  $\mu_M$ . As before, the same constructions may be done for a finitely generated group  $G$ , by taking  $M = K(G, 1)$ .

The *associated graded Lie algebra* of a group  $G$ ,  $\text{gr}^*(G) := \oplus_{k \geq 1} \Gamma_k G / \Gamma_{k+1} G$ , has Lie bracket  $[,]$  induced by the group commutator  $(,)$ . It follows that  $\text{gr}^*(G)$  is generated (as a Lie algebra) by  $\text{gr}^1(G)$ , and likewise for the rational associated graded Lie algebra,  $\text{gr}^*(G) \otimes \mathbb{Q}$ .

**Remark 2.3.** Recall the exact sequence

$$(2.5) \quad 0 \rightarrow K_G \rightarrow \wedge^2 H^1(G, \mathbb{Q}) \xrightarrow{\mu_G} DH_G^2 \rightarrow 0.$$

One also has the exact sequence

$$(2.6) \quad 0 \rightarrow N_G \rightarrow \wedge^2 \text{gr}^1(G) \otimes \mathbb{Q} \xrightarrow{\beta_G} \text{gr}^2(G) \otimes \mathbb{Q} \rightarrow 0,$$

where  $\beta_G$  denotes the Lie bracket. It follows from [20] that (2.6) is the vector space dual of (2.5), for any finitely generated group  $G$ .

**Remark 2.4.** For any given complex  $M$ , one may find a two-step nilpotent, torsion-free group  $G$ , such that  $\mathcal{R}_k^1(M) = \mathcal{R}_k^1(G)$ ,  $\forall k$ .

Set  $\pi = \pi_1(M)$ . Since the classifying map  $M \rightarrow K(\pi, 1)$  induces over  $\mathbb{Q}$  a cohomology isomorphism in degree one and a monomorphism in degree two, it

follows that  $\mu_M = \mu_\pi$ , hence  $\mathcal{R}_*^1(M) = \mathcal{R}_*^1(\pi)$ . Clearly,  $\text{gr}^{\leq 2}(\pi) \cong \text{gr}^{\leq 2}(\pi/\Gamma_3\pi)$ , as Lie algebras. In conclusion, the resonance varieties in degree one of a complex depend only on the third nilpotent quotient of its fundamental group:  $\mathcal{R}_*^1(M) = \mathcal{R}_*^1(\pi/\Gamma_3\pi)$ ; see Remark 2.3. Set  $N = \pi/\Gamma_3\pi$ . By construction,  $N$  is a finitely generated, two-step nilpotent group (i.e.,  $\Gamma_3N = \{1\}$ ). Finally, we may take  $G = N/\text{Tors}(N)$ . In this way, we may also achieve torsion-freeness, without changing  $\mu$ , since  $H^*(G, \mathbb{Q}) = H^*(N, \mathbb{Q})$ , as rings; see [13].

Plainly, the groups  $\pi$  and  $\pi/\pi''$  have the same third nilpotent quotient. We infer that in particular resonance in degree one depends only on the metabelian quotient of the fundamental group:  $\mathcal{R}_*^1(M) = \mathcal{R}_*^1(\pi/\pi'')$ . This is also true for characteristic varieties:  $\mathcal{V}_*^1(M) = \mathcal{V}_*^1(\pi/\pi'')$ , see e.g. [7]. However, it is no longer true, in general, that  $\mathcal{V}_1^1(G) = \mathcal{V}_1^1(G/\Gamma_3G)$ . Indeed, when  $G$  is a free group on  $n \geq 2$  generators, it is easy to see that  $\mathcal{V}_1^1(G) = (\mathbb{C}^*)^n$ , while  $\mathcal{V}_1^1(G/\Gamma_3G) = \{1\}$ , by Theorem 1.1.

**2.5. Poly-cyclic groups.** Let  $\mathcal{P}$  be a class of groups. Recall that a (length  $\ell$ ) finite normal series of a group  $G$  is a chain of subgroups,

$$(2.7) \quad 1 = G_\ell \subseteq \cdots \subseteq G_{i+1} \subseteq G_i \subseteq \cdots \subseteq G_1 = G,$$

with  $G_{i+1}$  normal in  $G_i$ ; if all subgroups  $G_i$  are normal in  $G$ , we speak about an invariant series. The group  $G$  is *poly- $\mathcal{P}$*  if it has a poly- $\mathcal{P}$  series (2.7), i.e.,  $G_i/G_{i+1} \in \mathcal{P}$ , for  $1 \leq i < \ell$ . If moreover the series is invariant, we call  $G$  *Poly- $\mathcal{P}$* . In this case, we have for each  $1 \leq i < \ell$  an extension with kernel in  $\mathcal{P}$ ,

$$(2.8) \quad 1 \rightarrow G_i/G_{i+1} \rightarrow G/G_{i+1} \rightarrow G/G_i \rightarrow 1.$$

It is well-known that the finitely generated nilpotent groups coincide with the Poly-cyclic groups for which all extensions (2.8) are central. Moreover, if  $G$  is finitely generated nilpotent and all lower central series quotients,  $\Gamma_j G/\Gamma_{j+1}G$ , are torsion-free, then  $G$  is Poly- $\mathbb{Z}$ , with all extensions central.

**Example 2.6.** Let  $A$  be a finitely generated non-trivial abelian group, and  $\alpha \in \text{Aut}(A)$ . We may form the semi-direct product extension,  $1 \rightarrow A \rightarrow A \rtimes_\alpha \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 1$ , where  $G := A \rtimes_\alpha \mathbb{Z}$  is finitely generated. Clearly,  $G$  is Poly-cyclic (Poly- $\mathbb{Z}$ ), if  $A$  is cyclic (respectively  $A = \mathbb{Z}$ ). Denote by  $t \in G$  the canonical lift of  $1 \in \mathbb{Z}$ . By the construction of  $G$ , it is easily seen that the length  $i+1$  commutator  $(t, (t, \dots (t, a) \dots))$  is equal to  $(\alpha - \text{id})^i(a)$ , for any  $a \in A$ . This remark may be used to show that, if  $\alpha = -\text{id}$  and  $A = \mathbb{Z}$  or  $A = \mathbb{Z}/k\mathbb{Z}$  (with  $k$  odd), then  $G$  is not nilpotent.

**Lemma 2.7.** *Assume in (2.1) that all groups are finitely generated. If either the extension is central, or the group  $K$  is finite, then  $\mathcal{V}_1^i(B) \subseteq \{1\}$ , for all  $i$ , implies that  $\mathcal{V}_1^i(G) \subseteq \{1\}$ , for all  $i$ .*

*Proof.* When the extension is central, one may use the same argument as in the proof of Theorem 2.2. (If  $\rho \in \mathbb{T}_G$  is not trivial on  $K$ , then the  $E^2$  page from (2.2) vanishes, since  $K$  is abelian.)

Pick  $1 \neq \rho \in \mathbb{T}_G$ . Assuming  $K$  is finite, we have  $H_+(K, \mathbb{C}_\rho) = 0$ ; see [2]. If  $\rho$  is non-trivial on  $K$ , we also have  $H_0(K, \mathbb{C}_\rho) = 0$ , and we are done. Otherwise,  $\rho = p^* \rho'$ , with  $1 \neq \rho' \in \mathbb{T}_B$ . Due to the finiteness of  $K$ ,  $H_+(K, \mathbb{C}_1) = 0$ . As explained before,  $H_0(K, \mathbb{C}_1) = \mathbb{C}_{\rho'}$ , over  $B$ . The spectral sequence (2.2) collapses, and  $H_*(G, \mathbb{C}_\rho) = E_{*0}^\infty = E_{*0}^2 = H_*(B, \mathbb{C}_{\rho'}) = 0$ .  $\square$

By induction on length, we obtain the following extension of Theorem 2.2. Note that the generalization is strict; see Example 2.6.

**Theorem 2.8.** *Let  $G$  be a Poly-cyclic group with the property that all extensions (2.8) with infinite kernel are central. Then  $\mathcal{V}_1^i(G) \subseteq \{1\}$ , for all  $i$ .*

**Lemma 2.9.** *Assume in (2.1) that  $K = \mathbb{Z}$  and  $B$  is finitely generated. If  $\mathcal{V}_1^1(G) \subseteq \{1\}$  and  $\mathcal{V}_1^i(B) \subseteq \{1\}$ , for all  $i$ , then the extension is central.*

*Proof.* The conjugation action of  $B = G/K$  on  $H_1(K, \mathbb{C}_1) = \mathbb{C}$  is encoded by a character  $\gamma \in \mathbb{T}_B$ . We have to show that  $\gamma = 1$ .

Pick any  $\rho' \in \mathbb{T}_B$ . In the spectral sequence (2.2) associated to  $\rho = p^* \rho'$ , we have

$$H_{>1}(\mathbb{Z}, \mathbb{C}_1) = 0, \quad H_0(\mathbb{Z}, \mathbb{C}_1) = \mathbb{C}_{\rho'} \quad \text{and} \quad H_1(\mathbb{Z}, \mathbb{C}_1) = \mathbb{C}_{\gamma \rho'},$$

over  $B$ . Suppose  $\gamma \neq 1$  and take  $\rho' = \gamma^{-1}$ . Then the  $E^2$ -page is concentrated on  $E_{s1}^2 = H_s(B, \mathbb{C}_1)$ , by our assumption on  $\mathcal{V}_1^*(B)$ . This implies that  $H_1(G, \mathbb{C}_\rho) = E_{01}^\infty = E_{01}^2 = H_0(B, \mathbb{C}_1) = \mathbb{C}$ . Consequently,  $1 \neq \rho \in \mathcal{V}_1^1(G)$ , contradicting the hypothesis.  $\square$

We are now in a position to show that (1.2) is a powerful property, which enables one to detect nilpotence in the class of Poly- $\mathbb{Z}$  groups. Compare with Example 2.6, case  $A = \mathbb{Z}$ .

**Theorem 2.10.** *Let  $G$  be a Poly- $\mathbb{Z}$  group. Then  $G$  is nilpotent if and only if  $\mathcal{V}_1^1(G) \subseteq \{1\}$ .*

*Proof.* We have to show that  $G$  must be nilpotent, if (1.2) holds. We induct on the length  $\ell$  of a poly- $\mathbb{Z}$  invariant series (2.7).

Note first that  $\mathcal{V}_1^1(G/G_i) \subseteq \mathcal{V}_1^1(G)$ , for all  $i$ . This follows easily by inspecting the spectral sequence (2.2) in low degrees, for the extension  $1 \rightarrow G_i \rightarrow G \rightarrow G/G_i \rightarrow 1$  and a character of  $G/G_i$ .

It follows that property (1.2) is inherited by the groups  $G/G_i$ , for  $1 \leq i < \ell$ . Clearly, they all are Poly- $\mathbb{Z}$ , so induction applies. Consider now the extensions (2.8), with kernel  $\mathbb{Z}$ . We know that  $\mathcal{V}_1^1(G/G_{i+1}) \subseteq \{1\}$ . We infer from Theorem 2.2 that  $\mathcal{V}_1^*(G/G_i) \subseteq \{1\}$ , since  $G/G_i$  is nilpotent, by induction. Lemma 2.9 tells us that the extension is central, hence  $G/G_{i+1}$  is nilpotent, too. This gives the desired nilpotence of  $G = G/G_\ell$ .  $\square$

### 3. ALEXANDER POLYNOMIAL AND METABELIAN GROUPS

**3.1. Alexander polynomial.** Let  $M$  be a connected complex, with finite 1-skeleton and fundamental group  $G = \pi_1(M, m_0)$ . Set  $G_{\text{ab}} := G/G'$  and  $G_{\text{abf}} := G_{\text{ab}}/\text{Tors}(G_{\text{ab}}) \cong \mathbb{Z}^n$ . Let  $p : X \rightarrow M$  be the Galois  $\mathbb{Z}^n$ -cover of  $M$  corresponding to the kernel of the canonical surjection,  $G \twoheadrightarrow G_{\text{abf}}$ . The finitely presented  $\mathbb{ZZ}^n$ -module  $A_G := H_1(X, p^{-1}(m_0); \mathbb{Z})$  is called the *Alexander module*, and depends only on  $G$ . Denote by  $\mathcal{E}_1(A_G) \subseteq \mathbb{ZZ}^n$  the first *elementary ideal* of the Alexander module, and let  $\Delta^G := \text{g.c.d.}(\mathcal{E}_1(A_G)) \in \mathbb{ZZ}^n$  be the Alexander polynomial.

If  $G$  is a finitely presented group (e.g., a finitely generated nilpotent group), everything may be computed in terms of a given finite presentation, as follows; see Fox [11]. Let  $M$  be the 2-complex associated to the presentation  $G = \langle x_1, \dots, x_m \mid w_1, \dots, w_s \rangle$ , with cellular chain complex  $(C_\bullet, \partial_\bullet)$ . Denote by  $(\widetilde{C}_\bullet, \widetilde{\partial}_\bullet)$  the equivariant cellular chain complex of the universal cover  $\widetilde{M}$ ,

$$(3.1) \quad \widetilde{C}_\bullet = \dots 0 \rightarrow \mathbb{Z}G \otimes C_2 \xrightarrow{\widetilde{\partial}_2} \mathbb{Z}G \otimes C_1 \xrightarrow{\widetilde{\partial}_1} \mathbb{Z}G \otimes C_0.$$

The  $m \times s$  matrix of  $\widetilde{\partial}_2$  may be computed by Fox differential calculus in the free group on  $x_1, \dots, x_m$ . It is equal to  $(\frac{\partial w_i}{\partial x_j})$ , modulo the defining relations of  $G$ . The  $1 \times m$  matrix of  $\widetilde{\partial}_1$  is simply  $(x_j - 1)$ , modulo defining relations. The Alexander module  $A_G$  is presented by

$$(3.2) \quad A_G = \text{coker} \left( (\mathbb{ZZ}^n)^s \xrightarrow{\mathbb{A}_G} (\mathbb{ZZ}^n)^m \right),$$

where  $\mathbb{A}_G := \mathbb{ZZ}^n \otimes_{\mathbb{Z}G} \widetilde{\partial}_2$ . Finally, one may also recover  $\partial_\bullet$  from (3.1), by taking the reduction of  $\widetilde{\partial}_\bullet$  modulo the augmentation ideal of  $G$ ,  $I_G \subseteq \mathbb{Z}G$ .

**Lemma 3.2.** *If  $\Delta^G(1) = 0$ ,  $b_1(G) \geq 2$ .*

*Proof.* By the very construction of  $\Delta^G$ , our assumption implies that  $\text{rk}(\mathbb{A}_G(1)) \leq s - 2$ , over  $\mathbb{Q}$ . The conclusion follows by resorting to (3.1).  $\square$

**3.3. Proof of Corollary 1.3.** We have to prove that  $\Delta^G \doteq 1$  (equality modulo units in  $\mathbb{C}\mathbb{Z}^n$ ), if  $G$  is finitely generated nilpotent. In other words, we must show that the zero set  $V(\Delta^G) \subseteq \text{Hom}(\mathbb{Z}^n, \mathbb{C}^*) = (\mathbb{C}^*)^n$  is empty.

It follows from [7, Proposition 2.4], via our Theorem 1.1, that  $V(\Delta^G) \subseteq \{1\}$ . Were  $V(\Delta^G)$  non-empty, we would have  $V(\Delta^G) = \{1\}$ , in particular  $n \geq 2$ , by virtue of Lemma 3.2.

Two cases may arise: either  $\Delta^G = 0$ , or  $\Delta^G \neq 0, 1$ . In the first situation,  $\dim V(\Delta^G) = n$ , and in the second  $\dim V(\Delta^G) = n - 1$ . In both cases, we arrive at a contradiction.

**Example 3.4.** Any non-zero constant may appear in Corollary 1.3. Indeed, Fox calculus applied to the standard presentation of  $G = \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$  shows that  $\Delta^G$  equals  $k$ .

**3.5. Proof of Theorem 1.2.** If  $G$  is metabelian (i.e., if  $G''$  is trivial), one has an extension  $1 \rightarrow B_{\text{ab}} \rightarrow G \rightarrow G_{\text{ab}} \rightarrow 1$ , where the Alexander invariant  $B_{\text{ab}} = G'/G''$  is endowed with the canonical  $\mathbb{Z}G_{\text{ab}}$ -module structure coming from conjugation in  $G$ . According to [13, Proposition I.4.1],  $G$  is nilpotent if and only if  $I \subseteq \sqrt{\text{ann } B_{\text{ab}}}$ , where  $I$  is the augmentation ideal of  $\mathbb{Z}G_{\text{ab}}$ . Since  $B_{\text{ab}}$  is torsion-free, by assumption, we are left with proving the above inclusion with  $\mathbb{C}$ -coefficients.

We also know that  $G_{\text{ab}} = G_{\text{abf}} = \mathbb{Z}^n$ . Therefore, the zero sets in  $(\mathbb{C}^*)^n$  of the elementary ideals  $\mathcal{E}_1(A_G)$  and  $\mathcal{E}_0(B_{\text{ab}})$  coincide, away from 1; see e.g. [7, Corollary 2.3]. It is also well-known that in this case the zero set of  $\mathcal{E}_1(A_G)$  coincides with  $\mathcal{V}_1^1(G)$ , away from 1 (see for instance [7, Proposition 2.4]). Putting things together, we infer from (1.2) that  $V(\mathcal{E}_0(B_{\text{ab}})) \subseteq \{1\} = V(I)$ , since  $I$  is generated by  $(t_i - 1)_{1 \leq i \leq n}$ .

A standard result in commutative algebra [8, pp. 511–513] says that the zero set  $V(\mathcal{E}_0(B_{\text{ab}}))$  equals  $V(\text{ann } B_{\text{ab}})$ . Thus, we have the inclusion  $V(\text{ann } B_{\text{ab}}) \subseteq V(I)$ . By Hilbert's Nullstellensatz,  $I \subseteq \sqrt{\text{ann } B_{\text{ab}}}$ .

The proof of Theorem 1.2 is complete.

**Remark 3.6.** Note that torsion-freeness is really necessary in the above theorem. Indeed, recall the group  $G = \mathbb{Z}/k\mathbb{Z} \rtimes_{\text{id}} \mathbb{Z}$  from Example 2.6, which is clearly finitely generated metabelian, but not nilpotent. However,  $\mathcal{V}_1^1(G) \subseteq \{1\}$  (use Lemma 2.7), the reason being the existence of torsion in  $G$ .

Having Theorem 2.10 in mind, we ought to point out that, in general, the groups to which the nilpotence test from Theorem 1.2 applies need not be Poly- $\mathbb{Z}$ . Indeed, it is easy to check by induction that the commutator subgroup  $G'$  must be finitely generated, if  $G$  is Poly- $\mathbb{Z}$ . On the other hand, it is equally easy to see that the finitely generated metabelian quotient  $G = \mathbb{F}/\mathbb{F}''$ , where  $\mathbb{F}$  is a free group



on  $n \geq 2$  generators, is torsion-free with torsion-free abelianization, but  $G'$  is not finitely generated.

Finally, note also that there are many finitely generated nilpotent groups with torsion-free lower central series quotients (hence, Poly- $\mathbb{Z}$ ), but not metabelian.

**3.7. A converse to Lemma 3.2.** Let  $G = \langle x_1, \dots, x_m \mid w_1, \dots, w_s \rangle$  be a finitely presented group. Set  $n = b_1(G)$ . We are going to view  $\Delta^G$  in  $\mathbb{k}[[x_1, \dots, x_n]]$ ,  $\mathbb{k}$  a field, by Magnus expansion,  $e : \mathbb{Z}\mathbb{Z}^n \rightarrow \mathbb{k}[[x_1, \dots, x_n]]$ . Here  $e$  is the ring homomorphism defined by sending each  $t_i$  to  $1 + x_i$ ; it clearly sends  $I := I_{\mathbb{Z}^n} = (t_i - 1)_i$  into the maximal ideal  $\mathfrak{m} = (x_i)_i$ . Set  $n_p := b_1(G, \mathbb{k})$ , noting that it depends only on  $p = \text{char}(\mathbb{k})$ , and that  $n_0 = n$ .

We first show how to find a *minimal* presentation of  $\mathbb{k}[[x_1, \dots, x_n]] \otimes_{\mathbb{Z}\mathbb{Z}^n} A_G$ .

**Lemma 3.8.** *Let  $G = \langle x_1, \dots, x_m \mid w_1, \dots, w_s \rangle$  be a finitely presented group, and  $\mathbb{k}$  be a characteristic  $p$  field. Then*

$$\widehat{A}_G := \mathbb{k}[[x_1, \dots, x_n]] \otimes_{\mathbb{Z}\mathbb{Z}^n} A_G = \text{coker} \left( \mathbb{k}[[x_1, \dots, x_n]]^t \xrightarrow{\overline{A}} \mathbb{k}[[x_1, \dots, x_n]]^{n_p} \right),$$

over  $\mathbb{k}[[x_1, \dots, x_n]]$ , where  $\overline{A} \equiv 0 \pmod{\mathfrak{m}}$ , and  $n_p - t = m - s$ .

*Proof.* We know from (3.2) that  $\widehat{A}_G = \text{coker}(\widehat{A}_G := \mathbb{k}[[x_1, \dots, x_n]] \otimes_{\mathbb{Z}\mathbb{Z}^n} \widehat{A}_G)$ , over  $\mathbb{k}[[x]] := \mathbb{k}[[x_1, \dots, x_n]]$ . Use linear algebra to find  $\mathbb{k}$ -vector space decompositions,  $C_2 \otimes \mathbb{k} = Z_2 \oplus B_1$  and  $C_1 \otimes \mathbb{k} = N_1 \oplus B_1$ , with respect to which  $\widehat{A}_G(1) = 0 \oplus \text{id}$ . It follows that  $\dim N_1 = n_p$  and  $\dim N_1 - \dim Z_2 = m - s$ .

Consider now the  $\mathbb{k}[[x]]$ -submodule  $\widehat{A}_G(\mathbb{k}[[x]] \otimes B_1)$  of the free module  $\mathbb{k}[[x]] \otimes C_1$ . Since the  $\mathfrak{m}$ -adic filtration of  $\mathbb{k}[[x]]$  is complete, one may look at the associated graded picture to infer from  $\widehat{A}_G(1) = 0 \oplus \text{id}$  that the above submodule is a free summand, with complement isomorphic to  $\mathbb{k}[[x]] \otimes N_1$ . The lemma follows.  $\square$

**Corollary 3.9.** *Let  $G$  be a finitely presented group and  $\mathbb{k}$  a characteristic  $p$  field. Set  $n = b_1(G, \mathbb{Q})$  and  $n_p = b_1(G, \mathbb{k})$ . Then*

$$\mathcal{E}_i(A_G) \subseteq \mathfrak{m}^{n_p - i}, \quad \text{if } i < n_p,$$

where  $\mathcal{E}_i$  denotes the  $i$ -th elementary ideal, and  $\mathfrak{m}$  is the maximal ideal of the formal power series ring  $\mathbb{k}[[x_1, \dots, x_n]]$ .

*Proof.* Due to the natural behaviour of elementary ideals under base change, we may use Lemma 3.8 to replace  $A_G$  by  $\text{coker}(\overline{A})$ . By adding trivial relations if necessary, we may also assume that  $m \leq s$ . If  $i < n_p$ ,  $\mathcal{E}_i(\widehat{A}_G)$  is generated by the  $(n_p - i)$ -minors of the minimal presentation matrix  $\overline{A}$ , and we are done.  $\square$

Following [7], we will say that the *Alexander ideal*  $\mathcal{E}_1(A_G)$  is *almost principal*, over a field  $\mathbb{k}$ , if

$$(3.3) \quad I^d \cdot (\Delta^G) \subseteq \mathcal{E}_1(A_G), \quad \text{in } \mathbb{k}G_{\text{abf}},$$

for some  $d \geq 0$ , where  $I$  denotes the augmentation ideal of  $G_{\text{abf}}$ .

**Example 3.10.** Property (3.3) holds for the following classes of groups: positive-deficiency groups  $G$  with  $b_1(G) \geq 2$ , for which  $d = 1$  (see [9]); fundamental groups of closed, connected, orientable 3-manifolds, where  $d = 2$  (see [16]).

**Proposition 3.11.** *Let  $G$  be a finitely presented group with  $n := b_1(G, \mathbb{Q}) > 0$ , and  $\mathbb{k}$  a characteristic  $p$  field. Set  $n_p := b_1(G, \mathbb{k})$ . If  $\mathcal{E}_1(A_G)$  is almost principal over  $\mathbb{k}$  and  $n_p > d + 1$ , then  $\Delta^G \in \mathfrak{m}^{n_p - d - 1}$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathbb{k}[[x_1, \dots, x_n]]$ . In particular,  $\Delta^G(1) = 0 \in \mathbb{k}$ .*

*Proof.* Corollary 3.9 guarantees that  $\mathcal{E}_1(A_G) \subseteq \mathfrak{m}^{n_p - 1}$ . Hence,  $x_1^d \cdot \Delta^G \in \mathfrak{m}^{n_p - 1}$ , by (3.3). Clearly, we may suppose that  $\Delta^G \neq 0$  in  $\mathbb{k}[[x]]$ , and  $d > 0$ . Take the initial term of  $\Delta^G$  in  $\mathbb{k}[[x]]$  to conclude that  $\Delta^G \in \mathfrak{m}^{n_p - d - 1}$ , as asserted.  $\square$

**Remark 3.12.** By resorting to Proposition 3.11 (over  $\mathbb{C}$ ) and Corollary 1.3, the proof of Theorem 1.4 is reduced to verifying the following claim.

A torsion-free, finitely generated nilpotent group  $G$ , with  $b_1(G) \leq 2$  and  $\text{def}(G) > 0$ , is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}^2$ .

See Example 3.10.

#### 4. MALCEV LIE ALGEBRAS

We will handle the case  $b_1(G) \leq 2$  by Malcev Lie algebra techniques.

**4.1. Associated graded and Malcev Lie algebras of groups.** A *Malcev Lie algebra* is a rational Lie algebra  $L$ , together with a descending, complete  $\mathbb{Q}$ -vector space filtration  $\{F_r L\}_{r \geq 1}$ , such that:  $F_1 L = L$ ;  $[F_r L, F_s L] \subseteq F_{r+s} L$ , for all  $r, s$ ; the associated graded Lie algebra,  $\text{gr}^*(L) := \bigoplus_{r \geq 1} F_r L / F_{r+1} L$ , is generated in degree 1. There is a natural Malcev Lie algebra  $E_G$ , associated to a group  $G$ . See [19, Appendix A] for details.

For example, if  $G$  is the free group on  $x_1, \dots, x_n$ ,  $E_G$  is the free Malcev Lie algebra  $\widehat{\mathbb{L}}(x_1, \dots, x_n)$ , that is, the degree completion of the free  $\mathbb{Q}$ -Lie algebra  $\mathbb{L}^*(x_1, \dots, x_n)$  graded by bracket length, endowed with the canonical filtration of formal series. If  $G = \langle x_1, \dots, x_m \mid w_1, \dots, w_s \rangle$ ,

$$(4.1) \quad E_G \cong \widehat{\mathbb{L}}(x_1, \dots, x_m) / \langle \langle r_1, \dots, r_s \rangle \rangle,$$

where  $\langle\langle\bullet\rangle\rangle$  denotes the closed Lie ideal generated by  $\bullet$ . The filtration of  $E_G$  comes from  $\widehat{\mathbb{L}}$ , and the Lie relators  $r_i$  are constructed from the corresponding group relators  $w_i$ , by Campbell-Hausdorff expansion. See [18].

A fundamental property of Quillen's construction  $E_G$  is the existence of a natural graded Lie algebra isomorphism,

$$(4.2) \quad \mathrm{gr}^*(E_G) \cong \mathrm{gr}^*(G) \otimes \mathbb{Q}.$$

In dual form, the Malcev Lie algebra of a finitely generated group  $G$  is nothing else but D. Sullivan's 1-minimal model of  $G$ . Consequently

**Theorem 4.2** ([20]). *If  $\varphi: G \rightarrow K$  is a morphism between finitely generated groups, inducing over  $\mathbb{Q}$  a cohomology isomorphism in degree 1 and a cohomology monomorphism in degree 2, then  $E_G \cong E_K$ , as filtered Lie algebras.*

**4.3. A second reduction.** The lemma below will be used to reduce the claim from Remark 3.12 to a statement about Malcev Lie algebras.

**Lemma 4.4.** *A finitely generated nilpotent, torsion-free group  $G$  is isomorphic to  $\mathbb{Z}^n$  if and only if  $E_G \cong E_{\mathbb{Z}^n}$ , as filtered Lie algebras.*

*Proof.* Assuming  $E_G \cong E_{\mathbb{Z}^n}$ , we infer from (4.2) that  $\mathrm{gr}^1(G) \otimes \mathbb{Q} = \mathbb{Q}^n$  and  $\mathrm{gr}^{>1}(G) \otimes \mathbb{Q} = 0$ . Therefore the groups  $\Gamma_i G / \Gamma_{i+1} G$  are finite, for  $i \geq 2$ . Since  $\Gamma_i G = 1$  for  $i \gg 0$ , we deduce that  $\Gamma_2 G$  is finite, hence trivial, by torsion-freeness. This means that  $G$  must be abelian, whence the result.  $\square$

**Remark 4.5.** Using the above lemma, we may suppose in Remark 3.12 that  $b_1(G) = 2$ , having to show that  $\mu_G \neq 0$ , in the notation from §§2.1.

Indeed, if  $b_1(G) = 0$  then  $E_G \cong E_{\{1\}}$ , by Theorem 4.2, hence  $G = \{1\}$ . But this cannot happen, since the deficiency of the trivial group is zero. If  $b_1(G) = 1$ , the same argument shows that  $G \cong \mathbb{Z}$ , as claimed. Finally, assume  $b_1(G) = 2$  and  $\mu_G \neq 0$ . Then Theorem 4.2, applied to the canonical morphism  $\varphi: G \rightarrow G_{\mathrm{abf}} \cong \mathbb{Z}^2$ , gives an isomorphism  $E_G \cong E_{\mathbb{Z}^2}$ . Invoking once more Lemma 4.4, we may thus complete the proof of the claim from Remark 3.12.

**4.6. Minimal Malcev Lie algebras.** We will need the following noncommutative analog of the *minimal* presentation constructed in Lemma 3.8.

**Proposition 4.7.** *Let  $E = \widehat{\mathbb{L}}(x_1, \dots, x_m) / \langle\langle r_1, \dots, r_s \rangle\rangle$  be a finitely presented Malcev Lie algebra. There is an isomorphism of filtered Lie algebras,*

$$E \cong \widehat{\mathbb{L}}(x'_1, \dots, x'_n) / \langle\langle r'_1, \dots, r'_t \rangle\rangle,$$

where  $r'_j \in F_2 \widehat{\mathbb{L}}(x')$ , for all  $j$ , and  $n - t = m - s$ .

*Proof.* Denote by  $X$  the  $\mathbb{Q}$ -vector space with basis  $\{x_1, \dots, x_m\}$ . Let  $Y$  be the  $\mathbb{Q}$ -vector space with basis  $\{y_1, \dots, y_s\}$ . Define a  $\mathbb{Q}$ -linear map,  $r : Y \rightarrow \widehat{\mathbb{L}}(X)$ , by  $r(y_j) = r_j$ , and set  $\bar{r} = \pi \circ r$ , where  $\pi : \widehat{\mathbb{L}}(X) \rightarrow \text{gr}^1(\widehat{\mathbb{L}}(X)) = X$  is the canonical projection. Choose vector space decompositions,  $Y = Z \oplus B$  and  $X = N \oplus B$ , such that  $\bar{r} = 0 \oplus \text{id}$ . Define a filtered Lie algebra map,  $f : \widehat{\mathbb{L}}(X) \rightarrow \widehat{\mathbb{L}}(X)$ , on the free generators, by:  $f(z) = z$ , for  $z \in N$ , and  $f(z) = r(z)$ , for  $z \in B$ .

We claim that  $f$  is a filtered Lie isomorphism. Indeed,  $\text{gr}^1(f) = \text{id}$ , by construction. Hence,  $\text{gr}^*(f)$  is onto, since  $\text{gr}^*(\widehat{\mathbb{L}}(X))$  is generated in degree one. By a dimension argument,  $\text{gr}^*(f)$  is an isomorphism. Our claim follows, by completeness of the Malcev filtration.

Interpreting  $f$  as a change of free generators, we obtain the desired minimal presentation, where  $n = \dim N$  and  $t = \dim Z$ .  $\square$

## 5. PROOF OF THEOREM 1.4

5.1. To finish the proof, we are left with verifying the following.

**Lemma 5.2.** *Let  $E = \widehat{\mathbb{L}}(x_1, x_2) / \langle \langle r \rangle \rangle$  be a one-relator Malcev Lie algebra, with  $r \in F_2 \widehat{\mathbb{L}}(x)$ . If  $\dim_{\mathbb{Q}} \text{gr}^*(E) < \infty$ , then  $r \not\equiv 0 \pmod{F_3 \widehat{\mathbb{L}}(x)}$ .*

Indeed, let us start with a finitely generated nilpotent, positive deficiency group  $G$ , and assume  $b_1(G) = 2$ , as in Remark 4.5. By (4.1),

$$E_G \cong \widehat{\mathbb{L}}(x'_1, \dots, x'_m) / \langle \langle r'_1, \dots, r'_{m-1} \rangle \rangle.$$

Moreover,  $\text{gr}^1(E_G) = \mathbb{Q}^2$ , see (4.2). Resorting to Proposition 4.7, we find a minimal Malcev Lie presentation,  $E_G \cong \widehat{\mathbb{L}}(x_1, x_2) / \langle \langle r \rangle \rangle$ , with  $r \in F_2 \widehat{\mathbb{L}}(x)$ . Since  $G$  is nilpotent, we also know that  $\dim_{\mathbb{Q}} \text{gr}^*(E_G) < \infty$ , again by (4.2).

Granting Lemma 5.2, we infer that  $\text{gr}^2(G) \otimes \mathbb{Q} = \text{gr}^2(E_G) = 0$ , since  $\mathbb{L}^2(x_1, x_2)$  is one-dimensional, generated by  $[x_1, x_2]$ . This in turn implies that  $\mu_G$  is an isomorphism, see Remark 2.3. In particular,  $\mu_G \neq 0$ , as claimed in Remark 4.5.

5.3. We embark now on the proof of Lemma 5.2. Clearly,  $r \neq 0$ , if  $\dim \text{gr}^*(E) < \infty$ , so  $r = r_d + \text{higher terms}$ , where  $0 \neq r_d \in \mathbb{L}^d(x_1, x_2)$  and  $d \geq 2$ . We have to show that  $d = 2$ .

By [15, §3], we know that  $\text{gr}^*(E) = \mathbb{L}^*(x_1, x_2) / \text{ideal}(r_d)$ , since  $r_d$  is an inert Lie element, in the sense of [12]. We have thus finally reduced the proof of Theorem 1.4 to the following assertion.

**Lemma 5.4.** *Let  $L^* = \mathbb{L}^*(x_1, x_2) / \text{ideal}(r_d)$  be a finite dimensional graded Lie algebra over  $\mathbb{Q}$ , where  $0 \neq r_d \in \mathbb{L}^d(x_1, x_2)$  and  $d \geq 2$ . Then necessarily  $d = 2$ .*

*Proof.* Let  $UL^* = \mathbb{T}^*(x_1, x_2)/\text{ideal}(r_d)$  be the universal enveloping algebra (where  $\mathbb{T}^*$  denotes the tensor algebra, graded by tensor length), with Hilbert series  $U(z)$ . By inertia, we have

$$(5.1) \quad U(z)^{-1} = 1 - 2z + z^d,$$

see [12, Théorème 2.4].

Set  $a_i = \dim_{\mathbb{Q}} L^i$ . By the Poincaré-Birkhoff-Witt theorem,

$$(5.2) \quad U(z)^{-1} = \prod_{i \geq 1} (1 - z^i)^{a_i},$$

where the infinite product is a polynomial, since  $\dim_{\mathbb{Q}} L^* < \infty$ , by assumption.

By comparing (5.1) and (5.2), we deduce that the polynomial  $1 - 2z + z^d$  is divisible by  $(1 - z)^2$ , since clearly  $a_1 = 2$ . This can only happen for  $d = 2$ .  $\square$

The proof of Theorem 1.4 is complete.

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